1. ECE 107 WEBSITE: http://cem01.ucsd.edu/courses/ece107
Problem 1

You are given a parallel plate transmission line made of perfect electrically conducting plates of width \( w = 1 \text{ cm} \) with the distance between the plates of \( d = 0.25 \text{ cm} \). The space between the plates is filled with vacuum. The transmission line is short-circuited at \( z = 0 \). It is excited by a source of frequency \( f = 1.8 \text{ GHz} \).

a) Find parameters characterizing the transmission line, i.e. \( R', L', G', C', \beta, Z_0, \lambda, u_p \).

b) The voltage at \( z_1 = -3 \text{ cm} \) is \( \tilde{V}(z_1) = 5e^{j/4} V \).

- Give expressions describing the voltage and current in the transmission line in the form of phasors and in the time domain representation.
- Find the reflection coefficient \( \Gamma \) at the point \( AA' \).
- Find the reflection coefficient \( \Gamma \) at the point \( BB' \).
Problem 2

Consider an infinitesimally thin ring with inner radius \( r_1 \) and outer radius \( r_2 \) that resides in the \( x-y \) plane and is centered at \( z=0 \) (Fig. 2(a)). The ring has a surface charge density

\[
\rho_s = \frac{\rho_0}{r},
\]

where \( r = \sqrt{x^2 + y^2} \).

a) Find the electric field \( \mathbf{E}(x=0, y=0, z) \) on the \( z \) axis.

Now suppose the ring is rotating counterclockwise (as viewed from above) about its center with a constant angular velocity \( \omega_0 \) (Fig. 2(b)).

b) Find the surface current distribution \( \mathbf{J}_s \) on the ring. (Hint: \( \mathbf{J}_s = \rho_s \mathbf{v} \), where \( \mathbf{v} \) is the linear velocity).

c) Find the magnetic field \( \mathbf{H}(x=0, y=0, z) \) on the \( z \) axis generated by the surface current density \( \mathbf{J}_s \).

![Fig. 2](image_url)

Here are some integrals you might find useful.

\[
\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\left( \frac{x}{a} \right) + C
\]

\[
\int \frac{x}{a^2 + x^2} \, dx = \frac{1}{2} \ln(a^2 + x^2) + C
\]

\[
\int \frac{x^2}{a^2 + x^2} \, dx = x - a \tan^{-1}\left( \frac{x}{a} \right) + C
\]

\[
\int \frac{1}{\sqrt{a^2 + x^2}} \, dx = \ln(x + \sqrt{a^2 + x^2}) + C
\]

\[
\int \frac{x}{\sqrt{a^2 + x^2}} \, dx = \sqrt{a^2 + x^2} + C
\]

\[
\int \frac{x^2}{\sqrt{a^2 + x^2}} \, dx = \frac{x}{a^2} \sqrt{a^2 + x^2} - \frac{x}{a} \ln(x + \sqrt{a^2 + x^2}) + C
\]

\[
\int \frac{x^2}{(a^2 + x^2)^{3/2}} \, dx = \frac{x}{a^2} \sqrt{a^2 + x^2} + C
\]

\[
\int \frac{x}{(a^2 + x^2)^{3/2}} \, dx = -\frac{1}{\sqrt{a^2 + x^2}} + C
\]

\[
\int \frac{x^2}{(a^2 + x^2)^{3/2}} \, dx = \ln(x + \sqrt{a^2 + x^2}) - \frac{x}{\sqrt{a^2 + x^2}} + C
\]
Problem 3

Consider an interface between two dielectric half-spaces with material parameters \( \varepsilon_1, \mu_0 \) and \( \varepsilon_2, \mu_0 \) (Fig. 3(a)). A perpendicularly (TE) polarized plane wave with amplitude \( E_0 \) is incident on this interface from medium 1 under an angle \( \theta \) with respect to the normal to the interface.

a) Consider the case where \( \theta \neq 0 \) (oblique incidence) and \( \varepsilon_1 \neq \varepsilon_2 \) (different half-spaces)
   - Write expressions for the incident, reflected, and transmitted electric and magnetic fields.
   - Write expressions for the incident, reflected, and transmitted power densities (Poynting vectors) including the magnitude and direction.
   - What are the surface currents and charges

b) Consider the case where \( \theta = 0 \) (normal incidence) and \( \varepsilon_1 \neq \varepsilon_2 \) (different half-spaces).
   - Give conditions for which the amplitude of the transmitted electric field is greater than that of the incident electric field. Give conditions for which the amplitude of the transmitted magnetic field is greater than that of the incident magnetic field.
   - For these cases, what are the ratios between the transmitted and incident power densities (is it greater or smaller than unity)? What are the relations between the incident, reflected and transmitted power densities?

c) Consider the case where \( \theta = 0 \) (normal incidence), \( \varepsilon_1 = \varepsilon_2 \) (identical half-spaces), but there is an infinitesimally thin sheet inserted on the interface between the half-spaces (Fig. 3(b)). Under an applied electric field, there is a surface current on the sheet that is given by \( \hat{z} \times J_s = \sigma_s \hat{z} \times E \), where \( \sigma_s \) is a given constant (of units \( \Omega^{-1} \)).
   - Find the reflection and transmission coefficients for this problem.
     Hint: Note that the total field in the left half-space \( (z < 0) \) is given by the sum of the incident and reflected field (i.e. \( E = E' + E''; z < 0 \)), whereas the total field in the right half-space \( (z > 0) \) is given only by the transmitted field (i.e. \( E = E'; z > 0 \)).
a) \( R' = 0 \text{ \( \Omega \text{m} \)} \), as plates are perfect conductors.

b) \( L' = \frac{\mu_0 d}{w} = \left(4\pi \times 10^{-7} \text{ H/m}\right) \left(\frac{0.25 \text{cm}}{1 \text{ cm}}\right) \)

\[ L' = 3.14 \times 10^{-7} \text{ H/m} \]

c) \( G' = \frac{\sigma w}{d} = 0 \text{ S/m} \) as the transmission line is filled with vacuum, \( \sigma = 0 \)

d) \( C' = \frac{\varepsilon_0 w}{d} = \left(8.85 \times 10^{-12} \text{ F/m}\right) \left(\frac{1 \text{cm}}{0.25 \text{cm}}\right) \)

\[ C' = 3.54 \times 10^{-11} \text{ F/m} \]

\[ \beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi \cdot 1.8 \times 10^9 \text{ Hz}}{3 \times 10^8 \text{ m/s}} \]

\[ \beta = 37.7 \text{ m}^{-1} \]

\[ Z_0 = \sqrt{\frac{L'}{C'}} \] for this lossless

\[ Z_0 = 94.2 \Omega \]

\[ \lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{ m/s}}{1.8 \times 10^9 \text{ Hz}} \]

\[ \lambda = 0.167 \text{ m} \]

\[ v_p = 3 \times 10^8 \text{ m/s} \]
b) \( \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \)

\( V(0) = V_0^+ + V_0^- = 0 \Rightarrow V_0^+ = -V_0^- \) for short circuit

\( V(z_1) = V(-3\text{ cm}) = V_0^+ e^{j\beta (3\text{ cm})} + V_0^+ e^{-j\beta (3\text{ cm})} = 5 e^{j\pi/4} V \)

\( V_0^+ \cdot (e^{j\beta (3\text{ cm})} + e^{-j\beta (3\text{ cm})}) = 5 e^{j\pi/4} \)

\( V_0^+ \cdot 2j \sin (\beta \cdot 3\text{ cm}) = 5 e^{j\pi/4} \)

\( V_0^+ = \frac{5}{2} \sin (\beta \cdot 3\text{ cm}) e^{j\pi/4} V \) as \( j = e^{j\pi/2} \)

\( = 2.26 e^{-j\pi/4} V = 1.6 - 1.6j \ V \)

\( \tilde{V}(z) = V_0^+ \left[ e^{-j\beta z} + e^{j\beta z} \right] = -2j V_0^+ \sin (\beta z) \)

\[ \tilde{V}(z) = 2.26 e^{-j\pi/4} \left[ e^{-j\beta z} - e^{j\beta z} \right] = -4.52 \sin (\beta z) e^{-j\pi/4} \]

\[ \tilde{I}(z) = \frac{V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z}}{Z_0} = \frac{V_0^+}{Z_0} \left( e^{-j\beta z} + e^{j\beta z} \right) \]

\[ = -\frac{V_0^+}{Z_0} \cdot 2 \cos (\beta z) \]

\[ \tilde{I}(z) = 0.024 e^{-j\pi/4} (e^{-j\beta z} + e^{j\beta z}) = 0.048 \cos (\beta z) e^{-j\pi/4} \]

\[ V(z,t) = \text{Re} \{ \tilde{V}(z) e^{j\omega t} \} \]

\[ V(z,t) = 2.26 \left[ \cos (\omega t - \beta z - \frac{\pi}{4}) - \cos (\omega t + \beta z - \frac{\pi}{4}) \right] = -4.52 \sin (\beta z) \cos (\omega t + \frac{\pi}{4}) \]

\[ \tilde{I}(z,t) = \text{Re} \{ \tilde{I}(z) e^{j\omega t} \} \]

\[ \tilde{I}(z,t) = 0.024 \left[ \cos (\omega t - \beta z - \frac{\pi}{4}) + \cos (\omega t + \beta z - \frac{\pi}{4}) \right] = 0.048 \cos (\beta z) \cos (\omega t - \frac{\pi}{4}) \]

\[ \Gamma_{AA'} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 \]

\( Z_L = 0 \) for short circuit
\[ \Gamma_{bb} = \frac{Z_{in}(z_1) - Z_o}{Z_{in}(z_1) + Z_o} \]

\[ -z_1 = l = 3 \text{ cm} \]

\[ \text{but } Z_L = 0 \quad (\text{short circuit}) \]

\[ Z_{in}(-l) = Z_o \frac{Z_L + Z_o \ j \tan (\beta l)}{Z_o + Z_L \ j \tan (\beta l)} \]

\[ Z_{in}(-l) = Z_o \ j \tan (\beta l) \]

\[ \Gamma_{bb} = \frac{Z_o \ j \tan (\beta l) - Z_o}{Z_o \ j \tan (\beta l) + Z_o} \]

\[ = \frac{j \tan (\beta l) - 1}{j \tan (\beta l) + 1} \]

\[ \Gamma_{bb} = e^{\gamma_0 l} \]

\[ = 0.637 + 0.770 \ j \]

\[ \rightarrow \text{ note that } |\Gamma| = 1, \text{ which is required by conservation of power. The same amount of power must be flowing left and right at } Z = 0 \text{ and } Z_1 = -3 \text{ cm, for a lossless TL} \]

\[ \text{Note also that} \]

\[ \Gamma_{bb} = \Gamma_{AA'} \ e^{-2\beta l} = \Gamma_{AA'} \ e^{2\beta z_1} \]

which is found by evolving the phase of the voltage waves from \( z_1 \) to \( Z = 0 \) and back to \( z_1 \).
2) a) \[ E = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho_s(r') (\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{|\hat{\mathbf{r}} - \hat{\mathbf{r}}'|^3} \, ds' \]

\[ \hat{\mathbf{r}} = \hat{\mathbf{z}} \]

\[ \hat{r}' = \hat{r}' \]

\[ \rho_s(r') = \frac{\rho_{so}}{r'} \]

\[ ds' = r' \, dr' \, d\phi \]

\[ \vec{E} = \frac{\rho_{so}}{4\pi \varepsilon_0} \int_0^{2\pi} d\phi \int_{r_i}^{r_2} \frac{1}{r'} \left( \frac{z \hat{\mathbf{z}} - r' \hat{\mathbf{r}}}{(z^2 + r'^2)^{3/2}} \right) r' \, dr' \]

but \( \vec{E} = E_z \hat{\mathbf{z}} \) by symmetry, so

\[ \vec{E} = \frac{\rho_{so}}{4\pi \varepsilon_0} \cdot 2\pi \int_{r_i}^{r_2} \frac{z \, dr'}{(\sqrt{z^2 + r'^2})^3} \]

\[ = \frac{\rho_{so}}{2 \varepsilon_0} \frac{z \, r_i}{z^2 \sqrt{z^2 + r_i^2}} \quad \bigg|_{r' = r_i} \]

\[ \vec{E} = \frac{\rho_{so}}{2 \varepsilon_0} \frac{z}{z_i} \left[ \frac{r_i}{\sqrt{z^2 + r_i^2}} - \frac{r_i}{\sqrt{z_i^2 + r_i^2}} \right] \hat{\mathbf{z}} \]

Alternative method:

\[ V = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho_s(r')}{|\hat{\mathbf{r}} - \hat{\mathbf{r}}'|} \, ds' = \frac{1}{4\pi \varepsilon_0} \cdot \rho_{so} \int \frac{1}{\sqrt{r^2 + z^2}} r' \, dr' \, d\phi' \]

\[ V = \frac{\rho_{so}}{2 \varepsilon_0} \ln \left( \frac{r_i + \sqrt{z_i^2 + r_i^2}}{r_i + \sqrt{z^2 + r_i^2}} \right) \]

\[ \vec{E} = -\nabla V, \quad \text{or} \quad E_z = -\frac{\partial}{\partial z} V = -\frac{\rho_{so}}{2 \varepsilon_0} \left[ \frac{z}{\sqrt{z^2 + r_i^2}} \cdot \frac{1}{r_i + \sqrt{z^2 + r_i^2}} \cdot \frac{r_i - \sqrt{z_i^2 + r_i^2}}{r_i - \sqrt{z^2 + r_i^2}} \right] \hat{\mathbf{z}} \]

\[ E_z = \frac{\rho_{so}}{2 \varepsilon_0} \left( \frac{r_i}{z_i} - \frac{r_i}{z_i} \right) \quad \text{Same result!} \]
b) \[ \frac{\vec{J}}{S} = \rho_s \vec{v}, \quad \vec{v} = rw \hat{\phi} \]

So \[ \frac{\vec{J}}{S} = \frac{\rho_s}{r} rw \hat{\phi} \]

\[ \frac{\vec{J}}{S} = \rho_s ow \hat{\phi} \]

\[ \vec{H} = \frac{1}{4\pi} \int \frac{\frac{\vec{J}}{S} \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3} dS' \]

\[ = \frac{\rho_s ow}{4\pi} \int \frac{\hat{\phi} \times (z \hat{\hat{z}} - r' \hat{\hat{r}})}{(z^2 + r'^2)^{3/2}} r' dr' d\phi' \]

\[ = \frac{\rho_s ow}{4\pi} \int \frac{z \hat{\hat{r}} + r' \hat{\hat{z}}}{(z^2 + r'^2)^{3/2}} r' dr' d\phi' \]

\[ \vec{H} = \vec{H}_z \hat{z} \quad \text{by symmetry} \]

\[ \vec{H} = \frac{\rho_s ow}{2} \int_{1}^{2} \frac{r'^2 dr'}{(z^2 + r'^2)^{3/2}} \hat{z} \]

\[ \frac{\vec{H}}{H} = \frac{\rho_s ow}{2} \left[ \ln \left( \frac{r_2 + \sqrt{r_2^2 + z^2}}{r_1 + \sqrt{r_1^2 + z^2}} \right) - \left( \frac{r_2}{\sqrt{r_2^2 + z^2}} - \frac{r_1}{\sqrt{r_1^2 + z^2}} \right) \right] \]
\( \vec{E}^i = \hat{y} E_0 e^{-j k_1 (x \sin \theta_i + z \cos \theta_i)} \)

\( \vec{H}^i = (-x \cos \theta_i + \hat{z} \sin \theta_i) \frac{E_0}{\eta_1} e^{-j k_1 (x \sin \theta_i + z \cos \theta_i)} \)

\( \eta_1 = \sqrt{\varepsilon_1 \mu_1} \)

**Reflected fields**

\( \vec{E}^r = \hat{y} \Gamma_{TE} E_0 e^{-j k_1 (x \sin \theta_i - z \cos \theta_i)} \), \( \Gamma_{TE} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \)

\( \vec{H}^r = (x \cos \theta_i + \hat{z} \sin \theta_i) \frac{E_0}{\eta_1} e^{-j k_1 (x \sin \theta_i - z \cos \theta_i)} \)

**Transmitted fields**

\( \vec{E}^t = \hat{y} \Gamma_{TE} E_0 e^{-j k_2 (x \sin \theta_i + z \cos \theta_i)} \), \( \gamma = \frac{2 \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \)

\( \vec{H}^t = (-x \cos \theta_i + \hat{z} \sin \theta_i) \frac{E_0}{\eta_2} e^{-j k_2 (x \sin \theta_i + z \cos \theta_i)} \), \( \eta_2 = \sqrt{\varepsilon_2 \mu_2} \)

We're given the incident frequency \( f \). We find the index of refraction in each medium to be

\( n_1 = \frac{c_0}{c_1} = \sqrt{\varepsilon_1 \mu_0} = \sqrt{\varepsilon_1} = \varepsilon_{r1} \), \( n_2 = \sqrt{\varepsilon_2 \mu_0} = \varepsilon_{r2} \)

\( k_1 = \frac{\omega}{c_1} = \frac{2\pi f}{c_1} = 2\pi f \sqrt{\varepsilon_1 \mu_0} \), \( k_2 = 2\pi f \sqrt{\varepsilon_2 \mu_0} \)

Find \( \theta_t \) using Snell's law

\( n_1 \sin \theta_i = n_2 \sin \theta_t \)

\( \theta_t = \sin^{-1} \left( \frac{n_1 \sin \theta_i}{n_2} \right) = \sin^{-1} \left( \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin \theta_i \right) \)

\( \rho_s = 0 \) and \( \vec{j}_s = 0 \) because the media are dielectrics

Time averaged power density is given by

\( \vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^* \)
Incident power density

\[ \vec{S}^i = \frac{1}{2} \vec{E}^i \times \vec{H}^i = \frac{1}{2} \frac{E_0^2}{\gamma_1} (\hat{x} \sin \theta_i + \hat{z} \cos \theta_i) \]

\[ \vec{S}^r = \frac{1}{2} \vec{E}^r \times \vec{H}^r = \frac{1}{2} \left| \vec{r} \right|^2 \frac{E_0^2}{\gamma_1} (\hat{x} \sin \theta_i - \hat{z} \cos \theta_i) \]

\[ \vec{S}^t = \frac{1}{2} \vec{E}^t \times \vec{H}^t = \frac{1}{2} \left| \vec{r} \right|^2 \frac{E_0^2}{\gamma_2} (\hat{x} \sin \theta_t + \hat{z} \cos \theta_t) \]

b) For the Electric field, the amplitude of the transmitted wave is

\[ E_0^t = \gamma E_0 = \frac{2 \eta_2}{\gamma_1 + \gamma_2} E_0 \]

So \( E_0^t > E_0 \) implies \( \gamma > 1 \), or \( 2 \eta_2 > \gamma_2 + \gamma_1 \)

or, since \( \gamma = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \)

\[ \eta_2 > \gamma_1 \]

\[ \varepsilon_1 > \varepsilon_2 \]

For the magnetic field,

\[ \frac{E_0^t}{\gamma_1 \gamma_2} = \gamma \frac{E_0}{\gamma_1 \gamma_2} \Rightarrow \frac{H_0^t}{\gamma_1} = \gamma \frac{H_0}{\gamma_2} \]

So \( H_0^t > H_0 \) implies \( \frac{\eta_2}{\gamma_1} < \frac{2 \eta_2}{\gamma_2 + \gamma_1} \)

\( \eta_2 + \gamma_1 < 2 \gamma_1 \)

\( \eta_2 < \gamma_1 \)

\[ \varepsilon_2 > \varepsilon_1 \]
For the electric field
\[
\frac{15 + 1}{15} = |\gamma|^2 \cdot \frac{\gamma_1}{\gamma_2}, \quad \text{where } \frac{\gamma_1}{\gamma_2} < 1
\]
\[
= \frac{\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2)^2} \cdot \frac{\gamma_2^2}{\gamma_2^2}
\]
\[
= \frac{4 \gamma_1}{(\gamma_1 + 1)^2}
\]
\[
\left| \frac{\gamma_1}{\gamma_2} \right| = \sqrt{1 + |\gamma|^2} + \frac{\gamma_1}{\gamma_2}
\]

In general,
\[
|\vec{s}^i| = |\vec{s}^p| + |\vec{s}^+|
\]

and
\[
1 = |\gamma|^2 + |\gamma|^2 \frac{\gamma_1}{\gamma_2}
\]

(C)
\[
\vec{E}^i = E_0 \hat{x} e^{-j k z} \quad \vec{H}^i = \frac{E_0}{\gamma} \hat{y} e^{-j k z}
\]
\[
\vec{E}^r = \gamma E_0 \hat{x} e^{j k z} \quad \vec{H}^r = -\gamma \frac{E_0}{\gamma} \hat{y} e^{j k z}
\]
\[
\vec{E}^+ = \gamma E_0 \hat{x} e^{-j k z} \quad \vec{H}^+ = \gamma \frac{E_0}{\gamma} \hat{y} e^{-j k z}
\]

Apply boundary conditions
First, the tangential electric field is continuous
\[
\vec{z} \times (\vec{E}^+ - (\vec{E}^i + \vec{E}^r)) \bigg|_{z=0} = 0, \quad \text{or } E^{i+} + E^{r+} = E^{++}
\]
this means
\[
1 + \Gamma = \gamma
\]
\[ \tilde{J}_s = \tilde{\Sigma} \times (\tilde{H}^i - (\tilde{H}^i + \tilde{H}^r)) \bigg|_{z=0} \]
\[ = \tilde{\Sigma} \frac{E_0}{\eta} \left(-\gamma - (-1 + \gamma)\right) \]
\[ = \tilde{\Sigma} \frac{E_0}{\eta} (1 - \gamma - \gamma) \]
\[ \hat{z} \times \frac{\tilde{J}_s}{\tilde{\Sigma}} = \hat{y} \frac{E_0}{\eta} (1 - \gamma - \gamma) = \sigma_0 \hat{z} \times \tilde{E} \bigg|_{z=0} \]

Using \[ \tilde{E} = \tilde{E}^+ \]
\[ \hat{y} \frac{E_0}{\eta} (1 - \gamma - \gamma) = \sigma_0 \hat{y} E_0 \hat{y} \]

so \[ 1 - \gamma - \gamma = \frac{\eta_0}{\eta_0} \sigma_0 \]

combine with \[ 1 + \gamma = \gamma \] gives

\[ 1 - \gamma - (1 + \gamma) = -2 \gamma = \eta_0 \sigma_0 (1 + \gamma) \rightarrow (\eta_0 \sigma_0 + 2) \gamma = -\eta_0 \sigma_0 \]

\[ \gamma = -\frac{\eta_0 \sigma_0}{2 + \eta_0 \sigma_0} \]

\[ \eta = \sqrt{\frac{\varepsilon_0}{\varepsilon_1}} \]

\[ \tau = \frac{\varepsilon_0}{2 + \eta_0 \sigma_0} \]