Signals and Systems problem for the Spring 2014 MS Exam in ECE

Suppose two continuous-time systems \( R \) and \( S \) are cascaded as shown in the diagram below. The system \( R \) has input \( x(t) \) and output \( y(t) \), and system \( S \) has input \( y(t) \) and output \( z(t) \). The assumptions in part (a) are independent from those of part (b), so do not mix them up.

(a) For the special case of \( \alpha = 1/2 \) and \( \tau = 250 \) microseconds, suppose the system \( S \) is an ideal low-pass filter on the interval \( \pm 1000 \) Hz. Find the magnitude of the Fourier transform of the signal \( z(t) \) if \( x(t) \) is a delta function. Carefully label the plot.

**SOLUTION:**

From the block diagram, the input/output relationship is \( y(t) = x(t) + \alpha x(t - \tau) \). When \( x(t) = \delta(t) \) we have \( X(\omega) = 1 \). Taking Fourier transforms gives

\[
H_R(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + \alpha e^{-j\omega\tau}
\]

\[
Z(\omega) = X(\omega) \cdot \frac{Y(\omega)}{X(\omega)} \cdot \frac{Z(\omega)}{H_R(\omega) \cdot H_S(\omega)}
\]

\[
\therefore |Z(\omega)| = |H_S(\omega)| \cdot \sqrt{(1 + \alpha \cos(\omega\tau))^2 + \alpha^2 \sin^2(\omega\tau)}
\]

\[
= \begin{cases} 
\sqrt{(1 + \alpha \cos(\omega\tau))^2 + \alpha^2 \sin^2(\omega\tau)} & \text{if } |\omega| \leq 2000\pi \\
0 & \text{else}
\end{cases}
\]

When \( f = 1000 \) Hz we have \( \omega = 2\pi f = 2000\pi \), so \( \omega\tau = (2000\pi)(250 \cdot 10^{-6}) = \pi/2 \), \( \cos(\omega\tau) = 0 \), and \( \sin(\omega\tau) = 1 \), which implies \( |Z(\omega)| = \sqrt{1 + \alpha^2} = \sqrt{5}/2 \approx 2.24/2 = 1.12 \). When \( \omega = 0 \), we have \( |Z(\omega)| = 1 + \alpha = 1.5 \). Also, \( |Z(\omega)| \) is an even function of \( \omega \), so it is symmetrical about \( \omega = 0 \).
(b) Suppose the system $S$ is such that $z(t) = x(t)$. Draw a block diagram of system $S$ Carefully label your blocks and signals.

**SOLUTION:**
From the block diagram, the input/output relationship is $y(t) = x(t) + \alpha x(t - \tau)$. Solving for $x(t)$ gives $x(t) = y(t) - \alpha x(t - \tau)$. This can be implemented in the block diagram shown below, where we changed notation $x \longrightarrow z$: