Math Question (equal weight each part)

Part 1

(a) Find the inverse of

\[ M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}. \]

Solution

The inverse may be found by creating a partitioned matrix

\[
\begin{bmatrix}
1 & 2 & | & 1 & 0 \\
3 & 2 & | & 0 & 1
\end{bmatrix}.
\]

Now apply the following row operations in succession

\[
3R_1 - R_2 \rightarrow R_2, \quad R_1 - \frac{1}{2}R_2 \rightarrow R_1 \quad \text{and} \quad \frac{1}{4}R_2 \rightarrow R_2.
\]

This yields

\[
\begin{bmatrix}
1 & 0 & | & -\frac{1}{2} & \frac{1}{2} \\
0 & 1 & | & \frac{3}{4} & -\frac{1}{4}
\end{bmatrix}
\]

so that the inverse of \( M \) is

\[
M^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}.
\]

(b) Use the inverse to find \( x \) if

\[ Mx = b \]

where

\[ b = \begin{bmatrix} 8 \\ 4 \end{bmatrix}. \]

Solution

Multiplying both sides by the inverse we have

\[
x = M^{-1}b = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.
\]

The solution may be verified by using back substitution.
Part 2

Find all solutions for \( x \) and \( \lambda \) to the following equation

\[ N x = \lambda x \]

where

\[ N = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \]

**Solution**

The characteristic equation is

\[(2 - \lambda)(3 - \lambda) - 2 = 0\]

or

\[\lambda^2 - 5\lambda + 4\]

which has roots \( \lambda = 1 \) and \( \lambda = 4 \). These are the eigenvalues.

The eigenvector \( v_1 \) for \( \lambda = 1 \) is given by

\[ \begin{bmatrix} 2 - \lambda_1 & 2 \\ 1 & 3 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \rightarrow x_1 + 2y_1 = 0 \rightarrow v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \]

The eigenvector \( v_2 \) for \( \lambda = 4 \) is given by

\[ \begin{bmatrix} 2 - \lambda_2 & 2 \\ 1 & 3 - \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \rightarrow x_2 - y_2 = 0 \rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

Part 3

Find all steady-state solutions for \( x(t) \) and \( \lambda \) to the following equation

\[ R x(t) = \lambda x(t) \]

where \( R \) is the derivative operator given by

\[ R = \frac{d^2}{dt^2} - 4 \frac{d}{dt} + 6 \]

**Solution** From your differential equations class, let the form of the steady-state solution be given by \( x(t) = e^{j\omega t} \) where \( \omega \) is an arbitrary frequency. Substituting this form into the equation and evaluating the derivatives, we have

\[ (-\omega^2 - j4\omega + 6) e^{j\omega t} = \lambda e^{j\omega t} \]

or

\[ \lambda(\omega) = -\omega^2 - j4\omega + 6. \]

Thus the eigenvalue for any value of \( \omega \) is determined by the transfer function for that value of \( \omega \). The corresponding eigenfunction is \( e^{j\omega t} \). This solution emphasizes that the eigenfunctions of a linear shift-invariant system are the complex exponentials with the eigenvalue given by the transfer function evaluated at the frequency \( \omega \).