Math Question-Solutions

Part 1

(i) Solve the following set of coupled differential equations

\[
\frac{dA(z)}{dz} = j(\gamma/2) B(z)
\]
\[
\frac{dB(z)}{dz} = j(\gamma/2) A(z)
\]

subject to the boundary conditions that \(A(0) = C\) and \(B(0) = 0\). (Note that \(j \doteq \sqrt{-1}\))

(ii) Sketch the solutions for \(|A(z)|^2\) and \(|B(z)|^2\) for a value of \(\gamma = \pi/4\).

Solution

Taking the derivative of each side and back substituting we obtain

\[
\frac{d^2A(z)}{dz^2} + \left(\frac{\gamma}{2}\right)^2 A(z) = 0.
\]

Using the boundary condition, this has solution \(A(z) = C \cos(\gamma z/2)\). Back solving we obtain \(B(z) = jC \sin(\gamma z/2)\). The plots are below for \(C = 1\) and \(\gamma = \pi/4\).

![Graphs of |A(z)|^2 and |B(z)|^2 for \(\gamma = \pi/4\)]

Part 2

A zero-mean joint Gaussian probability distribution is given by

\[
f_{xy}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right).
\]

(i) Derive the probability distribution \(f_z(z)\) for the random variable

\[z = x^2 + y^2,
\]

which is the squared-magnitude of the joint Gaussian probability distribution and sketch this distribution for \(\sigma^2 = 1\).
Solution

Using a cartesian-polar coordinate transformation given by \( x = r \cos \phi \) and \( y = r \sin \phi \) the joint-gaussian distribution in polar coordinates is

\[
f_{r\phi}(r, \phi) = \frac{r}{2\pi \sigma^2} \exp \left( -\frac{r^2}{2\sigma^2} \right),
\]

where \( \sin^2 \phi + \cos^2 \phi = 1 \) and the factor of \( r \) is from the differential area relationship \( dx \, dy = r \, dr \, d\phi \).

Now set \( r^2 = z \) so that \( dz = 2r \, dr \). The new joint pdf \( f_{z\phi}(z, \phi) \) is

\[
f_{z\phi}(z, \phi) = \frac{1}{4\pi \sigma^2} \exp \left( -\frac{z}{2\sigma^2} \right) dz \, d\phi.
\]

Integrating over \( \phi \), the marginal distribution for the squared-magnitude \( z = x^2 + y^2 \) is

\[
f_z(z) = \int_0^{2\pi} \frac{1}{4\pi \sigma^2} \exp \left( -\frac{z}{2\sigma^2} \right) d\phi
\]

\[
= \frac{1}{2\sigma^2} e^{-z/2\sigma^2}
\]

which is an exponential distribution with a mean value of \( 2\sigma^2 \). A plot is shown below

![Plot](plot.png)

(ii) Derive the probability distribution \( f_\phi(\phi) \) for the random variable

\[
\phi \doteq \tan^{-1} \left( \frac{y}{x} \right),
\]

which is the phase of the joint Gaussian distribution and sketch this distribution for \( \sigma^2 = 1 \).

Solution

The marginal phase distribution is determined by integrating \( f_{z\phi}(z, \phi) \) over \( z \). This yields

\[
f_\phi(\phi) = \frac{1}{2\pi}
\]

which is a uniform distribution.