Signals and Systems problem for the Spring 2013 MS Exam in ECE

Suppose a causal, linear, time-invariant system with input \( x(t) \) and output \( y(t) \) is described by the differential equation

\[
\frac{d^2y}{dt^2} - \frac{dx}{dt} + 5\frac{dy}{dt} = x - 6y
\]

Find the output \( y(t) \) when the input is \( x(t) = 2\delta(t - 1) - 4e^{-4t+4}u(t - 1) \), where \( \delta \) is the Dirac delta function and \( u \) is the unit step function.

**SOLUTION:** First, take Laplace transforms to find the system function \( H(s) \).

\[
s^2Y(s) + 5sY(s) + 6Y(s) = sX(s) + X(s)
\]

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{s + 1}{s^2 + 5s + 6} = \frac{s + 1}{(s + 2)(s + 3)}
\]

Let \( \hat{x}(t) = \delta(t) - 2e^{-4t}u(t) \). Then \( x(t) = 2\hat{x}(t - 1) \). We find the output \( \hat{y}(t) \) if \( \hat{x}(t) \) is the input and then use LTI properties to deduce the output due to \( x(t) \).

\[
\hat{X}(s) = 1 - \frac{2}{s + 4} = \frac{s + 2}{s + 4}
\]

\[
\hat{Y}(s) = H(s)\hat{X}(s) = \frac{s + 1}{(s + 2)(s + 3)} \cdot \frac{s + 2}{s + 4} = \frac{s + 1}{(s + 3)(s + 4)} = \frac{3}{s + 4} - \frac{2}{s + 3}
\]

\[
\therefore \hat{y}(t) = 3e^{-4t}u(t) - 2e^{-3t}u(t)
\]

\[
\therefore y(t) = 2\hat{y}(t - 1) = 6e^{-4(t-1)}u(t - 1) - 4e^{-3(t-1)}u(t - 1)
\]