Problem 41

(a) \( \hat{P}_{xx}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \left\{ \frac{1}{LU} \left\| \sum_{n=0}^{L-1} w(n) x_k(n) e^{-j2\pi fn} \right\|^2 \right\} \)

Where: 
- \( L \) = segment length
- \( w(n) = \) window function (e.g., Hann, Hanning, Kaiser-Bessel, etc.)
- \( x_k(n) = k^{\text{th}} \) segment where \( n \) is defined within the segment \( 0 \leq n < L-1 \)
- \( U = \frac{1}{L} \sum_{n=0}^{L-1} w^2(n) \)
- \( K = \) number of segments of length \( L \) (segments may overlap - if segments are adjacent with no overlap then \( N = LK \) when \( N \) is the total length of the data record)

(b) \( E\left[ \hat{P}_{xx}(f) \right] = \frac{1}{K} \sum_{k=0}^{K-1} \left\{ \frac{1}{LU} E\left[ \left\| \sum_{n=0}^{L-1} w(n) x_k(n) e^{-j2\pi fn} \right\|^2 \right] \right\} \)

Since \( E\left[ \left( \sum_{n=0}^{L-1} w(n) x_k(n) e^{-j2\pi fn} \right)^2 \right] \)

\[ = \sum_{n=0}^{L-1} \sum_{m=0}^{L-1} w(n) w(m) E\left[ x_k(n) x_k(m) \right] e^{-j2\pi fn} e^{j2\pi fm} \]

\[ = \sum_{n=0}^{L-1} \sum_{m=0}^{L-1} w(n) w(m) \delta(n-m) \]

\[ = \sum_{n=0}^{L-1} w(n)^2 \]

and \( LU = \sum_{n=0}^{L-1} w(n) \)

Then \( E\left[ \hat{P}_{xx}(f) \right] = \frac{\sum_{n=0}^{L-1} w(n)^2}{LU} \)

Note: 2-sided power spectrum
The use of the window function is to provide sidelobe control (minimize spectral leakage).

Small $K$ (large $L$) has the best frequency resolution since the DFT (Discrete Fourier Transform) is taken over a longer segment of the time series.

Large $K$ (small $L$) has the smallest variance since the variance of $\hat{P}_{xx}(\hat{f})$ decreases as the number of segment $K$ increases.

\[ x(n) = A \sin(2\pi f n) \] and goes through an integer number of cycles in $L$ points.

\[ X_k(f) = \frac{1}{L} \sum_{n=0}^{L-1} x(n) e^{-j2\pi fn} \]

where $k$ is a segment index and the segment length is $L$.

Recall $\sin(2\pi fn) = \frac{e^{j2\pi fn} - e^{-j2\pi fn}}{2j}$

Thus, substituting the expression for $x(n)$

\[ \left| X_k(f) \right| = \frac{A}{2} \sum_{n=0}^{L-1} \tilde{w}(n) \implies A = \frac{2}{\sum_{n=0}^{L-1} \tilde{w}(n)} \left| X_k(f) \right|^2 \]

Smasoid power $= \frac{A^2}{2} = \frac{2}{\left( \sum_{n=0}^{L-1} \tilde{w}(n) \right)^2} \left| X_k(f) \right|^2$

and is the same for all $k$.

Since $\hat{P}_{xx}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{LU} \left| X_k(f) \right|^2$

Then

Smasoid power $= \frac{A^2}{2} = \frac{2}{\left( \sum_{n=0}^{L-1} \tilde{w}(n) \right)^2} \left| X_k(f) \right|^2 = \frac{2}{\left( \sum_{n=0}^{L-1} \tilde{w}(n) \right)^2} \left\{ \frac{2}{\left( \sum_{n=0}^{L-1} \tilde{w}(n) \right)^2} \left\{ \frac{2}{\left( \sum_{n=0}^{L-1} \tilde{w}(n) \right)^2} \left\{ \frac{2}{\left( \sum_{n=0}^{L-1} \tilde{w}(n) \right)^2} \right\} \right\} \right\}$

\[ \hat{P}_{xx}(f) \] since $LU = \sum_{n=0}^{L-1} \tilde{w}(n)$
Problem #2

(a)

\[
x(n) = w(n) - \sum_{i=1}^{p} a_i x(n-i)
\]

\[
H(z) = \frac{1}{\sum_{i=0}^{p} a_i z^{-i}} \\
= \frac{z^p}{\sum_{i=0}^{p} a_i z^{-i}}
\]

\(a_0 = 1\)

(b)

\[
P_{xx}(f) = \frac{1}{W} \left| H(z) \right|^2 \quad z = e^{-j2\pi f}
\]

\[
= \frac{1}{W} \left| \sum_{i=0}^{p} a_i e^{-j2\pi fi} \right|^2 \quad a_0 = 1
\]
(c) One-step forward linear predictor of $x(n)$

$$x(n)$$

\[ \begin{array}{c}
\vdots \\
z_1^p \\
\vdots \\
z_n^p \\
\sum \\
\sum \\
\sum \\
a_1^p \\
q_2^p \\
q_p^p \\
\end{array} \]

\[ e_p(n) \]

\[ -x_{cn} \]

Problem: \[ \min_{a_p^p} E[e_p^2(n)] \]

Define \[ a_p^p = \begin{bmatrix} a_1^p \\ a_2^p \\ \vdots \\ a_p^p \end{bmatrix} \]

\[ x_{cn} = \begin{bmatrix} x(n-1) \\ x(n-2) \\ \vdots \\ x(n-p) \end{bmatrix} \]

\[ \phi(m) = E[x(n)x(n+m)] = \phi(-m) \]

\[ \phi = E[x(n)x^*(n)] = \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(p) \end{bmatrix} \]

\[ \Phi = E[x_{cn}x_{cn}^T] = \begin{bmatrix} \phi(0) & \ldots & \phi(p-1) \\ \vdots & \ddots & \vdots \\ \phi(p-1) & \ldots & \phi(0) \end{bmatrix} \]

Since $x(n)$ is wide sense stationary, $\Phi$ is Toeplitz and is completely defined by its upper row $\phi(m)$, $m = 0, 1, \ldots, p-1$. Since $x(n)$ is real, $\Phi$ is symmetric, $\Phi = \Phi^T$. 
\[ e_p(n) = x(n) + \sum_{i=1}^{p} a_i^p x(n-i) \]
\[ = x(n) + a^p x^-(n) \]
\[ e_p^2(n) = (x(n) + a^p x^-(n))^2 \]
\[ = (x(n) + a^p x^-(n)) (x(n) + x^-(n) a^p) \]
\[ = x^2(n) + z a^p x(n) x^-(n) + a^p x^{-T}(n) x^{-T}(n) a^p \]
\[ E[e_p^2(n)] = \nabla x^2 + z a^p g + a^p \mu a^p \]

Minimizing \( E[e_p^2(n)] \) with respect to \( a^p \) leads to
\[ 0 = g + \mu a^p \text{ or } \mu a^p = -g \]

Solving for \( a^p \):
\[ a^p = -\mu^{-1} g \]

(d) Forward prediction error power
\[ E_p = \min_{a^p} E[e_p^2(n)] = \nabla x^2 + z a^p g + a^p \mu (-\mu^{-1} g) \]
\[ = \nabla x^2 + a^p g = \Phi(\omega) + a^p \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(p) \end{bmatrix} \]
From the solution in (c), we have:

$$ \frac{\partial^p}{\partial t^p} \phi = -q $$

Adding $[\phi(1), \phi(2), \ldots, \phi(p)]^T$ to both sides yields:

$$
\begin{bmatrix}
\phi(1) \\
\phi(2) \\
\vdots \\
\phi(p) \\
\end{bmatrix}

\begin{bmatrix}
\phi(0) \\
\phi(1) \\
\vdots \\
\phi(p-1) \\
\end{bmatrix}

= 
\begin{bmatrix}
1 \\
a_1^p \\
a_2^p \\
\vdots \\
a_p^p \\
\end{bmatrix}

\begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
$$

Lastly, augmenting the above with the expression for $E_p$:

$$
\begin{bmatrix}
\phi(0) \\
\phi(1) \\
\vdots \\
\phi(p) \\
\end{bmatrix}

\begin{bmatrix}
\phi(0) \\
\phi(1) \\
\vdots \\
\phi(p-1) \\
\phi(p) \\
\end{bmatrix}

= 
\begin{bmatrix}
1 \\
a_1^p \\
a_2^p \\
\vdots \\
a_p^p \\
\end{bmatrix}

\begin{bmatrix}
E_p \\
0 \\
\vdots \\
0 \\
0 \\
\end{bmatrix}
$$