Math Question

(You can attempt all three parts. A passing grade is at least 50% for two out of the three parts.)

Part 1

Find the solution of the initial value problem

\[
\frac{dy}{dt} - 2ty = 1 \quad y(0) = 1
\]

Note: \( \int_0^t e^{-\tau^2} d\tau = (\sqrt{\pi}/2)\text{erf}(t) \) where \text{erf}(t) is the error function.

Solution

This is a first order-differential equation with a time-varying coefficient \(-2t\). To solve this equation, we first determine the integrating factor \(\mu(t)\), which is given by

\[
u(t) = \exp\left(-\int t \, dt\right) = e^{-t^2}.
\]

The integrating factor is defined such that

\[
\frac{du}{dt} \frac{1}{u} = -2t.
\]

Substitute this expression into the original equation and multiply both sides by \(u(t)\). The left side is then \( u \frac{dy}{dt} + y \frac{du}{dt} = d(y)/dt \) so that

\[
\frac{d}{dt} (u(t)y(t)) = u(t).
\]

Integrating both sides and dividing through by \(u(t)\) yields

\[
y(t) = e^{t^2} \left[ \int_0^t e^{-\tau^2} d\tau' + c \right].
\]

Applying the initial condition \(y(0) = 1\), yields \(c = 1\). Using the integral given in the problem statement, the final solution is

\[
y(t) = e^{t^2} \left[ \frac{\sqrt{\pi}}{2} \text{erf}(t) + 1 \right].
\]

Part 2

Find the general solution of the following matrix equation

\[
\begin{bmatrix}
6 & 5 & 4 \\
3 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
15 \\
6
\end{bmatrix}
\]
Solution

This is a inhomogeneous matrix equation. The complete solution consists of the sum of a particular solution and a homogeneous solution. From inspection, the particular solution is

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

The homogeneous solution is a vector

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
\]

that satisfies the following set of homogeneous equations

\[
\begin{align*}
6y_1 + 5y_2 + 4y_3 &= 0 \\
3y_1 + 2y_2 + y_3 &= 0
\end{align*}
\]

The complete solution is the sum of the homogeneous part and the inhomogeneous part

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_2
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} + \begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
\]

Part 3

A uniform random variable \(X\) is defined over the interval \([0,2]\). A second independent random variable \(Y\) has a probability density function \(P_Y(y) = 2e^{-2y}u(y)\) where \(u(y)\) is a unit-step function. Find and sketch the probability density function \(P_Z(z)\) for the random variable \(Z = X + Y\).

Solution

The probability density function of the sum of two independent random variables is the convolution of the individual probability density functions. The pdf for a uniform random variable is a rect function with a height that is equal to the reciprocal of the interval. For an interval of two, the height is then 1/2. Evaluating the convolution, there are two intervals. For \(z < 2\), we have

\[
P_Z(z) = \int_{0}^{z} e^{-2z'}dz' = \frac{1}{2} \left( 1 - e^{-2z} \right) \quad z < 2
\]

where the factor of 2 for \(P_Y(y)\) cancels the factor of 1/2 for the rect function. For \(z \geq 2\), we have

\[
P_Z(z) = \int_{z-2}^{z} e^{-2z'}dz' = \frac{1}{2} e^{-2z} \left( e^4 - 1 \right) \quad z \geq 2
\]
A plot is shown below