Problem 1: $x[n]$ is a zero mean, real, white Gaussian process. So $E(x[n]) = 0$ and $E(x[n]x[m]) = \sigma^2 \delta[n - m]$.

a) The periodogram estimate of the power spectrum is given by

$$
\hat{R}(e^{j\omega}) = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-j\omega n}
$$

b) Examining the mean of the Periodogram

$$
E(\hat{R}(e^{j\omega})) = \frac{1}{N} E \left[ \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \right]^2
$$

$$
= \frac{1}{N} E \left[ \left( \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \right) \left( \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \right)^* \right]
$$

$$
= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[n]x[m]e^{-j\omega m}e^{-j\omega m}
$$

$$
= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sigma^2 \delta[n - m]e^{-j\omega(n-m)} = \sigma^2
$$

It is an unbiased estimate, a property that can be shown to be true in general asymptotically.

$$
\text{Var}(\hat{R}(e^{j\omega})) = E(\hat{R}(e^{j\omega})\hat{R}(e^{j\omega})) - E(\hat{R}(e^{j\omega}))E(\hat{R}(e^{j\omega})) = E(\hat{R}(e^{j\omega})\hat{R}(e^{j\omega})) - \sigma^4. \quad (1)
$$

Examining the first term

$$
E(\hat{R}(e^{j\omega})\hat{R}(e^{j\omega})) = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} E(x[n]x[m]x[l]x[k])e^{-j\omega n}e^{j\omega m}e^{-j\omega l}e^{j\omega k}
$$

$$
= \frac{\sigma^4}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} \delta[n - m]\delta[l - k]e^{-j\omega n}e^{j\omega m}e^{-j\omega l}e^{j\omega k}
$$

$$
= T_1 + T_2 + T_3
$$

$$
T_1 = \frac{\sigma^4}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} \delta[n - m]\delta[l - k]e^{-j\omega n}e^{j\omega m}e^{-j\omega l}e^{j\omega k}
$$

$$
= \sigma^4 \left[ \frac{1}{N} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} \delta[l - k]e^{-j\omega l}e^{j\omega k} \right]
$$

$$
= \sigma^4
$$

Note that $T_1$ is cancelled by the second term in Eq. 1. Now we examine term $T_2$

$$
T_2 = \frac{\sigma^4}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} \delta[n - l]\delta[m - k]e^{-j\omega n}e^{j\omega m}e^{-j\omega l}e^{j\omega k}
$$

$$
= \sigma^4 \left[ \frac{1}{N} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} \delta[l - k]e^{-j\omega l}e^{j\omega k} \right]
$$

$$
= \sigma^4 \left[ \frac{1}{N} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} e^{-j2\omega l} \right]^2
$$

$$
= \sigma^4 \left[ \frac{1}{N} \frac{1 - e^{j2\omega N}}{1 - e^{j2\omega}} \right]^2
$$

$$
= \sigma^4 \frac{1}{N} \frac{1 - e^{j2\omega N}}{1 - e^{j2\omega}}
$$

Similar to $T_1$,

$$
T_3 = \sigma^4.
$$

Hence the Variance in the Periodogram estimate is

$$
\text{Var}(\hat{R}(e^{j\omega})) = \sigma^4 + \sigma^4 \frac{1}{N} \frac{1 - e^{j2\omega N}}{1 - e^{j2\omega}}
$$

$c)$ The estimate is unbiased. However the variance is not zero and does not converge to zero even asymptotically. It is not a consistent estimate
Problem 2:

a) The power spectrum is given

\[ R(e^{j\omega}) = \frac{1}{|A(e^{j\omega})|^2} = \frac{1}{|1 - 1.5e^{-j\omega} + 1.25e^{-j2\omega} - .5e^{-j3\omega} + .25e^{-j4\omega}|^2} \]

b) The best predictor of \( x[n] \) based on \( x[n-1], ..., x[n-10] \) is given

\[ \hat{x}[n] = 1.5x[n-1] - 1.25x[n-2] + .5x[n-3] - .25x[n-4] \]

The results follows from the orthogonality principle. The error is \( w[n] \) which is orthogonal to the data \( x[n], ..., x[n-10] \). The mean squared error is \( P_4 = E(w^2[n]) = 1 \).

c) The best predictor of order 3 is given by

\[ \hat{x}[n] = -a_1x[n-1] - a_2x[n-2] - a_3x[n-3] \]

The predictor coefficients can be found by using the Levinson-Durbin in reverse as follows

\[
\begin{bmatrix}
1 \\
-1.5 \\
1.25 \\
-.5 \\
.25
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
a_1 \\
a_2 \\
a_3 \\
0
\end{bmatrix}
+ .25
\begin{bmatrix}
a_3 \\
a_2 \\
a_1 \\
1
\end{bmatrix}
\]

This results in \( a_1 = -\frac{22}{15}, a_2 = 1, \) and \( a_3 = -\frac{2}{15} \). The mean squared error is \( P_3 = P_4/(1-.25^2) = \frac{16}{15} \).

d) The backward linear prediction of order \( M \) consists of predicting \( x[n-M] \) based on \( M \) future values \( x[n], ..., x[n-M+1] \), i.e.

\[ \hat{x}[n-M] = -\sum_{k=0}^{M-1} b_k^*x[n-k]. \]

The cost function is the mean squared error and the predictor coefficients are chosen to minimize \( E(|x[n-M] - \hat{x}[n-M]|^2) \). Based on the connection between forward linear prediction and backward linear prediction \( b_3 = a_1^* = -\frac{22}{15}, b_2 = a_2^* = 1, \) and \( b_1 = a_3^* = -\frac{2}{15} \). The mean squared error is same as the forward linear prediction of order 3 which is \( P_3 = \frac{16}{15} \).