UCSD ECE 45 Preparedness Test Solutions

1. A complex number z can be written in rectangular form as $z = a + jb$ or in phasor form as $z = |z|\angle\theta$. To convert between these different forms, we have the relationships:

$$a = |z|\cos\theta$$
$$b = |z|\sin\theta$$

and

$$|z| = \sqrt{a^2 + b^2}$$
$$tan(\theta) = \frac{b}{a}$$

So we have:

(a) $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\theta = 45°$

(b) $r = \sqrt{3^2 + 0^2} = 3$ and $\theta = 0°$

(c) $r = \sqrt{0^2 + (-2)^2} = 2$ and $\theta = 270°$

(d) $r = \sqrt{(-12)^2 + 3^2} = 3\sqrt{17}$ and $\theta = 165.96°$

2. It is usually easier to add/subtract complex numbers in rectangular form and to multiply/divide them in phasor form. For complex numbers $z_1 = a + jb = |z_1|\angle\theta_1$ and $z_2 = c + jd = |z_2|\angle\theta_2$, we have:

$$z_1 + z_2 = a + jb + c + jd = (a + c) + j(b + d)$$
$$z_1 - z_2 = a + jb - (c + jd) = (a - c) + j(b - d)$$
$$z_1 * z_2 = |z_1||z_2|\angle(\theta_1 + \theta_2)$$
$$\frac{z_1}{z_1} = \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2)$$

Also, for a complex number $z = a + jb = |z|\angle\theta$, the complex conjugate z^* is given by

$$z^* = a - jb = |z|\angle-\theta$$

So we have:

(a) $(4+3j)-(2-6j) = 2+9j = \sqrt{85}\angle 77.47°$

(b) $(1+2j)(4+6j) = (\sqrt{5}\angle 63.4°)(2\sqrt{13}\angle 56.3°) = 2\sqrt{65}\angle 119.7°$

(c) $(1+2j)(4-6j) = (\sqrt{5}\angle 63.4°)(2\sqrt{13}\angle -56.3°) = 2\sqrt{65}\angle 7.1°$

(d) $\dfrac{(2+4j)}{(6-7j)} = \dfrac{2\sqrt{5}\angle 63.43°}{\sqrt{85}\angle -49.4°} = \dfrac{2}{\sqrt{17}}\angle 112.83°$

(e) $\dfrac{(1+2j)+(3+4j)}{(2-3j)-4} = \dfrac{4+6j}{-2-3j} = \dfrac{2(2+3j)}{-1(2+3j)} = -2 = 2\angle 180°$

(f) $((1+2j)(2+3j))^* = ((\sqrt{5}\angle 63.4°)(\sqrt{13}\angle 56.3°))^* = (\sqrt{65}\angle 119.7°)^*$
$= \sqrt{65}\angle -119.7°$

3. To show (a), note that cos(-x) = cos(x) and sin(-x) = -sin(x) and plug in Euler's formula:

$$\frac{e^{jx}+e^{-jx}}{2} = \frac{cos(x)+jsin(x)+cos(-x)+jsin(-x)}{2}$$

$$= \frac{cos(x)+jsin(x)+cos(x)-jsin(x)}{2} = cos(x)$$

Similarly, to show (b):

$$\frac{e^{jx}-e^{-jx}}{2j} = \frac{cos(x)+jsin(x)-cos(-x)-jsin(-x)}{2j}$$

$$= \frac{cos(x)+jsin(x)-cos(x)+jsin(x)}{2j} = sin(x)$$

4. For these problems, we can use the representations of sine and cosine found in the previous problem.

(a) $f(t) = 1 + cos(t) + sin(2t+90°) = 1 + \dfrac{e^{jt}+e^{-jt}}{2} + \dfrac{e^{j(2t+90°)}-e^{-j(2t+90°)}}{2j}$

We can simplify this further by noting that $e^{j90°} = j$ and $e^{-j90°} = -j$:

$$f(t) = 1 + \frac{e^{jt}}{2} + \frac{e^{-jt}}{2} + \frac{e^{j2t}e^{j90°}}{2j} - \frac{e^{-j2t}e^{-j90°}}{2j}$$

$$f(t) = 1 + \frac{e^{jt}}{2} + \frac{e^{-jt}}{2} + \frac{e^{j2t}}{2} + \frac{e^{-j2t}}{2}$$

(b) First we use the trigonometric identity $cos^2(2t) = \frac{cos(4t)+1}{2}$. Using this, we find:

$$f(t) = \frac{cos(4t)+1}{2} + sin(3t) = \frac{e^{j4t}+e^{-j4t}}{4} + \frac{1}{2} + \frac{e^{j3t}-e^{-j3t}}{2j}$$

$$f(t) = \frac{1}{2} + \frac{e^{j4t}}{4} + \frac{e^{-j4t}}{4} + \frac{e^{j3t}}{2j} - \frac{e^{-j3t}}{2j}$$

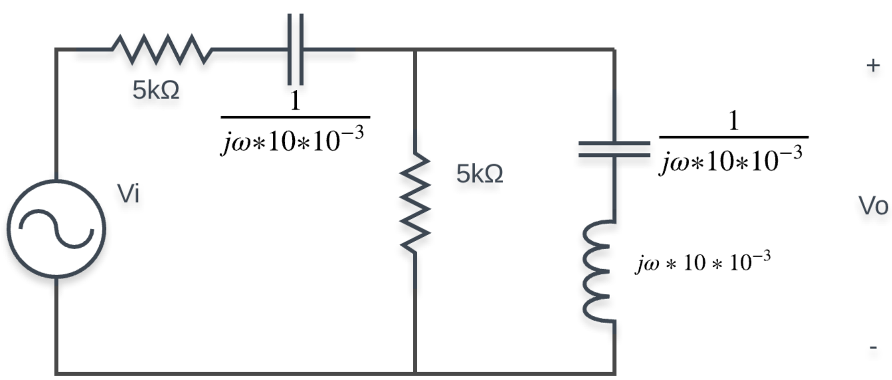
Alternately, we can just use the complex representation of sine and cosine:

$$f(t) = (\frac{e^{j2t} + e^{-j2t}}{2})^2 + \frac{e^{j3t} - e^{-j3t}}{2j} = \frac{e^{j4t} + 1 + 1 - e^{-j4t}}{4} + \frac{e^{j3t} - e^{-j3t}}{2j}$$

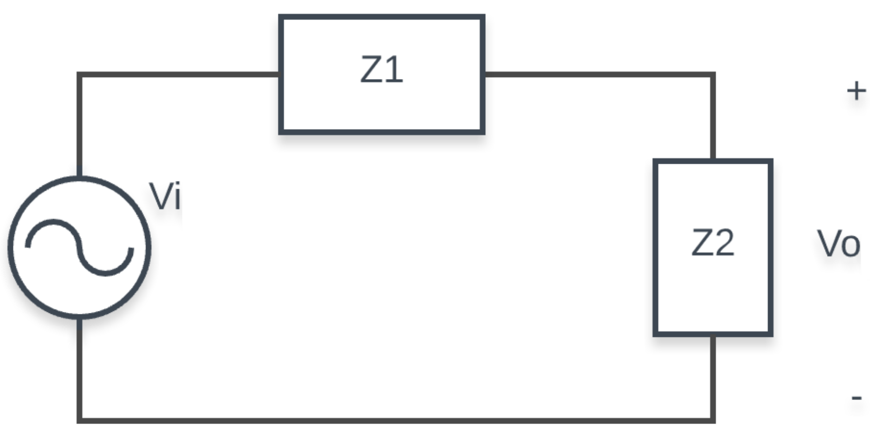
$$f(t) = \frac{1}{2} + \frac{e^{j4t}}{4} + \frac{e^{-j4t}}{4} + \frac{e^{j3t}}{2j} - \frac{e^{-j3t}}{2j}$$

As can be seen, both of these solution methods give the same answer (as we would expect).

5. First, let's redraw this circuit in the phasor domain. Doing so, we find the circuit:



Next, we simplify the circuit to the form below by finding the equivalent impedances Z_1 and Z_2.



We find Z_1 by noting that the horizontal 5kΩ resistor is in series with the 10mF capacitor:

$$Z_1 = 5000 + \frac{1}{j\omega * 10 * 10^{-3}}$$

Now we find Z_2 by noting that the vertical 5kΩ resistor is in parallel with the series combination of the 10mF capacitor and the 10mH inductor:

$$Z_2 = \frac{(\frac{1}{j\omega*10*10^{-3}} + j\omega * 10 * 10^{-3}) * 5000}{\frac{1}{j\omega*10*10^{-3}} + j\omega * 10 * 10^{-3} + 5000}$$

With these two impedances, we can find Vo:

$$V_o = V_i \frac{Z_2}{Z_1 + Z_2}$$

(a) For $v_i(t) = cos(100t)$, we have:

$$Z_2 = \frac{(\frac{1}{j100*10*10^{-3}} + j100 * 10 * 10^{-3}) * 5000}{\frac{1}{j100*10*10^{-3}} + j100 * 10 * 10^{-3} + 5000} = \frac{(-j + j) * 5000}{-j + j + 5000} = 0$$

Therefore, $V_o = 0$ so $V_o(t) = 0$

(b) For $v_i(t) = cos(10^6 t)$, we have:

$$Z_2 = \frac{(\frac{1}{j10^6*10*10^{-3}} + j10^6 * 10 * 10^{-3}) * 5000}{\frac{1}{j10^6*10*10^{-3}} + j10^6 * 10 * 10^{-3} + 5000} = \frac{(-j * 10^{-4} + j * 10^4) * 5000}{-j * 10^{-4} + j * 10^4 + 5000}$$

$$\approx \frac{j * 5 * 10^7}{j * 10^4 + 5000} = \frac{j * 50000}{5 + j10}$$

And

$$Z_1 = 5000 + \frac{1}{j * 10^6 * 10 * 10^{-3}} \approx 5000$$

Therefore we have

$$V_o \approx \frac{\frac{j*50000}{5+j10}}{\frac{j*50000}{5+j10} + 5000} = \frac{j * 50000}{25000 + j * 100000} = \frac{j2}{1 + j4} = \frac{j2 + 8}{17}$$

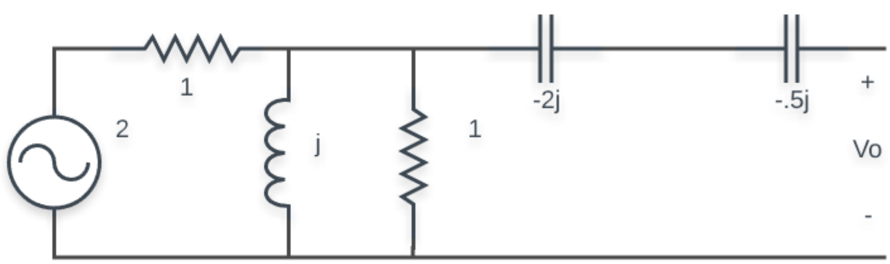
Thus we find our final answer by taking the phasor V_o and converting back to the time domain to find

$$V_o(t) = -\frac{2}{17}sin(10^6t) + \frac{8}{17}cos(10^6t)$$

(c) As $\omega \to \infty$, capacitors begin to look like shorts while inductors begin to look like open circuits. As a result, the circuit appears like a voltage divider between the two $5k\Omega$ resistors. Therefore we find $V_o \to \frac{1}{2}$ so we have $V_o(t) \to \frac{1}{2}cos(\omega t)$.

6. The easiest way to analyze this circuit is to use superposition. First, examine the contribution of the voltage source.

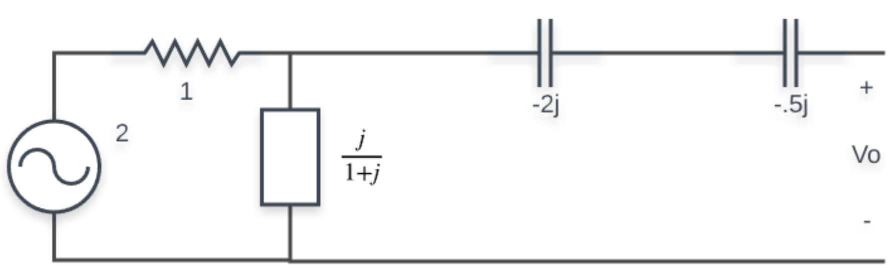
The voltage source has a frequency of $\omega_1 = 1$. Based on this frequency, we have the following circuit in the phasor domain:



The inductor is in parallel with the vertical 1 ohm resistor. These two in parallel have the equivalent impedance:

$$Z_{eq} = \frac{1 * j}{1 + j} = \frac{j}{1 + j}$$

Using this equivalent impedance, we can simplify the circuit further to:



Now, we can find $V_{o,v}$, the component of V_o due to the voltage source, using voltage division:

$$V_{o,v} = 2\frac{Z_{eq}}{Z_{eq}+1} = 2\frac{\frac{j}{1+j}}{\frac{j}{1+j}+1} = 2\frac{j}{1+j+j} = \frac{2j}{1+2j}$$

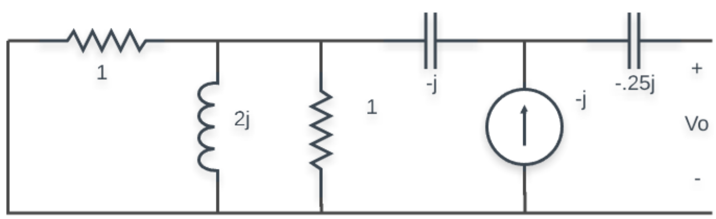
We can simplify this further by multiplying by the complex conjugate:

$$V_{o,v} = \frac{2j(1-2j)}{(1+2j)(1-2j)} = \frac{2j+4}{1+4} = \frac{2j+4}{5}$$

Now, we find the contribution to $V_o(t)$ from the voltage source in the time domain (recalling the frequency of the voltage source was $\omega_1 = 1$) by transforming from the phasor domain back into the time domain:

$$V_{o,v}(t) = \frac{4}{5}cos(t) - \frac{2}{5}sin(t)$$

After analyzing the contribution to V_o from the voltage source, we must consider the current source. The current source has a frequency of $\omega_2 = 2$. Based on this frequency, we have the following circuit in the phasor domain:



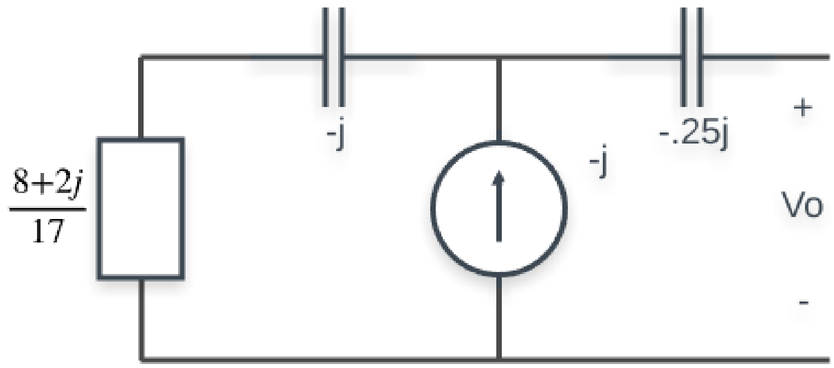
The two resistors are in parallel with the inductor. We can combine these to find an equivalent impedance of:

$$Z_{eq} = (\frac{1}{1} + \frac{1}{1} + \frac{1}{2j})^{-1} = (\frac{2j+2j+1}{2j})^{-1} = \frac{2j}{1+4j}.$$

Multiplying by the complex conjugate, we find

$$Z_{eq} = \frac{2j(1-4j)}{(1+4j)(1-4j)} = \frac{8+2j}{17}$$

This gives the circuit:



Since no current flows through the rightmost capacitor, the contribution to V_o from the current source, $V_{o,i}$ is found as:

$$V_{o,i} = -j(-j + Z_{eq}) = -j(-j + \frac{8+2j}{17}) = -j\frac{8-15j}{17} = \frac{-15-8j}{17}$$

With this we find the contribution in the time domain due to the current source (recall that the frequency $\omega_2 = 2$):

$$V_{o,i}(t) = -\frac{15}{17}cos(2t) + \frac{8}{17}sin(2t)$$

The total $V_o(t)$ is the sum of the contribution from the voltage source and the contribution from the current source, so:

$$V_o(t) = V_{o,v}(t) + V_{o,i}(t) = \frac{4}{5}cos(t) - \frac{2}{5}sin(t) - \frac{15}{17}cos(2t) + \frac{8}{17}sin(2t)$$

7. To find these summations, we need to use the relation

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

In the special case where $|a| < 1$, we have

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

as $a^\infty \rightarrow 0$. So we have:

(a) $\sum_{n=0}^{20} (\frac{1}{2})^n = \frac{1-(\frac{1}{2})^{21}}{1-\frac{1}{2}} = 2(1-(\frac{1}{2})^{21})$

(b) $\sum_{n=0}^{\infty} (\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2$

(c) $\sum_{n=1}^{\infty}\frac{1}{3}(\frac{1}{3})^n = \frac{1}{3}(\sum_{n=1}^{\infty}(\frac{1}{3})^n) = \frac{1}{3}(\sum_{n=0}^{\infty}(\frac{1}{3})^n - 1) = \frac{1}{3}(\frac{1}{1-\frac{1}{3}} - 1) = \frac{1}{3}(\frac{3}{2} - 1) = \frac{1}{6}$

8. To solve these problems, we need to use the following rules for exponents:

$$x^n(x^k) = x^{n+k}$$
$$\frac{x^n}{x^k} = x^{n-k}$$
$$(x^n)^k = x^{nk}$$

So we have:

(a) $x^2(x^3) = x^5$

(b) $\sqrt{x}x^4 = x^{1/2}x^4 = x^{4.5}$

(c) $x^{-2}(x^{0.7}) = x^{-1.3}$

(d) $\frac{x^5}{x^{10}} = x^{-5}$

(e) $((x^4)^3)^2 = (x^{12})^2 = x^{24}$

(f) $(x^5)^{4!} = (x^5)^{24} = x^{120}$

9. To solve these problems, we need to use the following log rules:

$$log_N(a) + log_N(b) = log_N(ab)$$
$$log_N(a) - log_N(b) = log_N(a/b)$$
$$log_N(a^k) = k \times log_N(a)$$
$$log_N(N) = 1$$
$$N^{log_N(a)} = a$$

So we have:

(a) $log_{10}(5) + log_{10}(3) = log_{10}(15) \rightarrow x = 15$

(b) $log_5(3) - log_5(5) = log_5(3/5) \rightarrow x = 3/5$

(c) $ln(6^8) = 8ln(6) \rightarrow x = 8$

(d) $log_{10}(x) = 5 \rightarrow 10^{log_{10}(x)} = 10^5 \rightarrow x = 10^5$

(e) $ln(3^x) = xln(3) = 7 \rightarrow x = \frac{7}{ln(3)}$

(f) $ln(x^3) = 3ln(x) = 7 \rightarrow ln(x) = 7/3 \rightarrow e^{ln(x)} = x = e^{7/3}$

10. (a) $\int cos(t)dt = sin(t) + c$

(b) $\int_0^t cos(t)dt = sin(t) - sin(0) = sin(t)$

(c) $\int \frac{5}{x}dx = 5\int \frac{1}{x} = 5ln(|x|) + c$

(d) $\int e^x dx = e^x + c$

(e) $\int e^{jx} dx = \dfrac{e^{jx}}{j} + c = -je^{jx} + c$

(f) For this integral, we need to first notice that the answer is a function of t. Also,

$$x(\tau)e^{j\tau} = \begin{cases} e^{j\tau} & -3 < \tau < 3 \\ 0 & else \end{cases}$$

So from $-\infty < t < -3$, we have $\int_{-\infty}^{t} 0 d\tau = 0$

From $-3 < t < 3$ we have $\int_{-\infty}^{t} x(\tau)e^{j\tau}d\tau = \int_{-3}^{t} e^{j\tau}d\tau = \frac{e^{jt} - e^{-j3}}{j} = -j(e^{jt} - e^{-j3})$

Lastly from $3 < t < \infty$ we have $\int_{-\infty}^{t} x(\tau)e^{j\tau}d\tau = \int_{-3}^{3} e^{j\tau}d\tau = -j(e^{3j} - e^{-3j})$

Therefore our final solution is $\int_{-\infty}^{t} x(\tau)e^{j\tau}d\tau = \begin{cases} 0 & -\infty < t < -3 \\ -j(e^{jt} - e^{-j3}) & -3 < t < 3 \\ -j(e^{3j} - e^{-3j}) & 3 < t < \infty \end{cases}$

(g) $\int_0^y xe^{-x^2}dx = \dfrac{-e^{-x^2}}{2}|_0^y = \dfrac{-e^{-y^2}}{2} - \dfrac{-1}{2} = \dfrac{1 - e^{-y^2}}{2}$

(h) To compute this integral, we need to use the partial fractions technique. We want to find A and B such that

$$\frac{2x+4}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$$

This gives us the equation

$$2x + 4 = A(x + 4) + B(x + 3)$$

We can find A by evaluating this equation at x = -3:

$$2(-3) + 4 = A(-3 + 4) \rightarrow A = -2$$

We can find B by evaluating this equation at x = -4:

$$2(-4) + 4 = B(-4 + 3) \rightarrow B = 4$$

So now we can rewrite our integrand as

$$\frac{2x+4}{(x+3)(x+4)} = \frac{-2}{x+3} + \frac{4}{x+4}$$

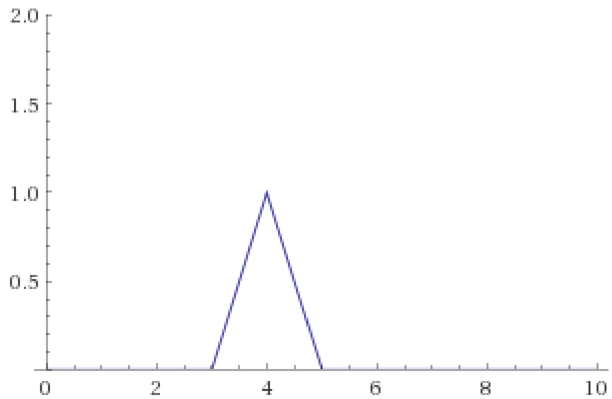
This gives us:

$\int_{-\frac{y}{2}}^{\frac{y}{2}} \dfrac{2x+4}{(x+3)(x+4)} = \int_{-\frac{y}{2}}^{\frac{y}{2}} \dfrac{-2}{x+3} + \int_{-\frac{y}{2}}^{\frac{y}{2}} \dfrac{4}{x+4}$

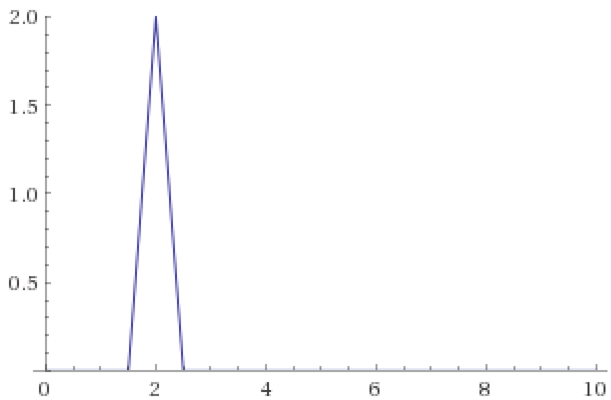
$= -2ln(|y/2 + 3|) + 2ln(|-y/2 + 3|) + 4ln(|y/2 + 4|) - 4ln(|-y/2 + 4|)$

11. The graphs are shown below:

(a) $y(t-2)$:



(b) $2y(2t-2)$:



(c) $\frac{1}{2}y(\frac{-t}{3}+4)$: